FUN WITH QUADRATIC EQUATIONS: ROOTS

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Level: HS algebra (squares and square roots)

A critical juncture in everyone's experience with learning algebra occurs when it is time to derive the roots of a quadratic equation. The equations for the roots themselves are complicated enough, but the usual and lengthy derivation, involving the rabbit-out-of-the-hat technique of "completing the square," is for many folks the exit ramp in their algebraic journey, the point where their intended college major, if any, changes to recreation management.

In 2019, an alternative, much simpler derivation was presented by Po-Shen Loh of Carnegie Mellon University (https://arxiv.org/abs/1910.06709). He does not claim that the approach is original, but scrutiny of math history has so far not revealed any demonstration of the approach.

The essential derivation is given here. However, the Loh derivation assumes two preliminary results that I think should be shored up before proceeding with the feature presentation.

The equation for a quadratic function can be represented as:

$$y = Ax^2 + Bx + C \ .$$

Here x and y are real numbers, and the equation tells how to compute y from any particular value of x. The shape of the function is a curve known as a parabola. The quantities A, B, and C are real-valued constants, where A must be nonzero, or the equation is not quadratic. The A, B, and C values prescribe the location and appearance of the parabola on a graph. For instance, Figure 1 illustrates four different parabolas formed by altering the values of A, B, and C.

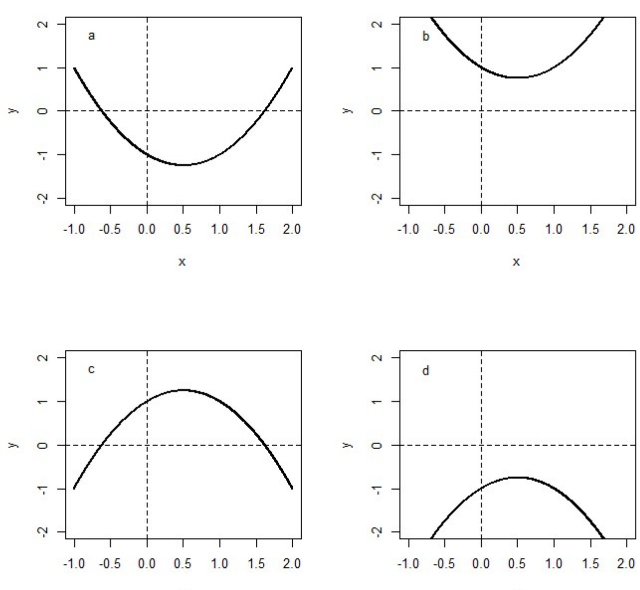

Figure 1. Four quadratic functions.
(a) $A = 1, B = -1, C = -1$. (b) $A = 1, B = -1, C = 1$.
(c) $A = -1, B = 1, C = 1$. (d) $A = -1, B = 1, C = -1$.

The value of A tells whether the parabola "opens upward" (like a bowl) or "opens downward" (like an upside-down bowl): if A is positive, the parabola is like a bowl (concave up), and if A is negative, the parabola is like an upside-down bowl (concave down). The value of C is the y-axis intercept: the value of y where the parabola curve cuts through the y-axis when x has the value 0. The value of B will turn out to be related to the value of x where the top (maximum) or bottom (minimum) of the parabola occurs. The roots of a quadratic equation, when they can be expressed as real numbers, are the two values of values of x where the curve cuts the x-axis when the value of y is 0. If the roots do not exist as real numbers, the curve is entirely above or below the x-axis.

The first preliminary result we need deals with something that the Loh derivation assumes but might not be so obvious at first sight. Suppose a, b, and c are three other real numbers. If the quantities given by $Ax^2 + Bx + C$ and $ax^2 + bx + c$ are equal for all values of x, what can be said about the values of A, B, C, a, b, and c?

A preliminary result needed for establishing a main result is typically called a "lemma." So, we have:

Lemma 1. If $Ax^2 + Bx + C = ax^2 + bx + c$ for all real values of x, where A, B, C, a, b, and c are all real numbers, then $A = a$, $B = b$, and $C = c$.

Proof. If indeed

$$Ax^2 + Bx + C = ax^2 + bx + c$$

for all real values of x, then we can equate the expressions for particular values of x. Suppose we evaluate the expressions at $x = 0$. Instantly we see that

$$C = c\ .$$

So now we have $Ax^2 + Bx + C = ax^2 + bx + C$, or, subtracting C from both sides,

$$Ax^2 + Bx = ax^2 + bx\ .$$

Now we insert $x = 1$ to get

$$A + B = a + b\ .$$

Also insert $x = -1$ to get

$$A - B = a - b\ .$$

Add the first and second expressions together to get $2A = 2a$, or

$$A = a\ .$$

And now we have

$$Ax^2 + Bx = Ax^2 + bx .$$

We go after B and b by subtracting Ax^2 from both sides to get

$$Bx = bx ,$$

and then dividing both sides by any nonzero value of x, producing the last part of our lemma,

$$B = b .$$

The above lemma is a special case of a result applying to all polynomials: if two polynomials of degree n are equal for all values of x, then their coefficients of like powers of x are equal.

We also need a second preliminary result for Loh's derivation. It may be obvious, but it should be formally stated:

Lemma 2. If a and b are any real numbers such that $a < b$, and $m = (a + b)/2$ is the arithmetic average of a and b, then $a = m - \frac{b-a}{2}$ and $b = m + \frac{b-a}{2}$.

In words, a and b can be written respectfully as their average minus something and their average plus something, where that something is half the distance between a and b. **Proof** follows easily by substituting $(a + b)/2$ for m in the lemma expressions and cancelling terms.

We will also need to recall two algebraic properties involving squares and square roots:

(i) $v^2 - w^2 = (v + w)(v - w)$.

(ii) $\sqrt{(v/w)} = \sqrt{v}/\sqrt{w}$.

With these results in hand, we are now ready for the Loh derivation of the roots of a quadratic equation. We could call the result a "theorem":

Theorem. The quadratic equation given by $y = Ax^2 + Bx + C$, where A is nonzero, has two real roots, that is, two values of x that satisfy $Ax^2 + Bx + C = 0$, given by

$$x_1 = \frac{-B-\sqrt{B^2-4AC}}{2A} ,$$

$$x_2 = \frac{-B+\sqrt{B^2-4AC}}{2A} ,$$

provided $B^2 > 4AC$. If $B^2 = 4AC$, then there is only one real root given by

$$x_1 = -\frac{B}{2A} ,$$

representing a parabola that just touches the horizontal axis at one point (the value of x where the parabola is maximized or minimized). If $B^2 < 4AC$, the quadratic equation has no real roots.

Proof. If $Ax^2 + Bx + C = 0$, then we can write $x^2 + \left(\frac{B}{A}\right)x + \left(\frac{C}{A}\right) = 0$. If A were zero, prohibiting that division, then we would not be dealing with a quadratic equation. Rewrite the quadratic equation as

$$\begin{aligned} y = \; & A(x - x_1)(x - x_2) = A[x^2 - (x_1 + x_2)x + x_1 x_2] \\ & = Ax^2 - A(x_1 + x_2)x + Ax_1x_2 \; . \end{aligned} \tag{1}$$

where x_1 and x_2 are constants that because of Lemma 1 must satisfy

$$-(x_1 + x_2) = \frac{B}{A}, \tag{2}$$

$$x_1 x_2 = \frac{C}{A}. \tag{3}$$

If $y = 0$, then x_1 and x_2 are seen in eq. (1) to be two values of x where the quadratic equation is zero, that is, x_1 and x_2 are roots of the quadratic equation. Arbitrarily let x_1 be the smaller root. From eq. (2) we have

$$\frac{(x_1+x_2)}{2} = -\frac{B}{2A}.$$

So, $-B/(2A)$ is the arithmetic average of x_1 and x_2 ! (Exclamation for excitement not for factorial.) That means, by Lemma 2, x_1 and x_2 are equidistant from $-B/(2A)$ and can be written as

$$x_1 = -\frac{B}{2A} - \frac{x_2 - x_1}{2}, \tag{4}$$

$$x_2 = -\frac{B}{2A} + \frac{x_2 - x_1}{2}. \tag{5}$$

Now, in eq. (3) we have the product of x_1 and x_2 . In light of eqs. (4) and (5) above, we can write the product as

$$x_1 x_2 = \left(-\frac{B}{2A} - \frac{x_2 - x_1}{2}\right)\left(-\frac{B}{2A} + \frac{x_2 - x_1}{2}\right).$$

Apply algebraic property (i) to the right side of the above expression to get

$$x_1 x_2 = \frac{B^2}{4A^2} - \left(\frac{x_2 - x_1}{2}\right)^2.$$

From eq. (3) we then find that

$$\frac{B^2}{4A^2} - \left(\frac{x_2 - x_1}{2}\right)^2 = \frac{C}{A} ,$$

or

$$\left(\frac{x_2 - x_1}{2}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A} .$$

By taking the square root of both sides, it appears that we have written the quantity $(x_2 - x_1)/2$ in eqs. (4) and (5) entirely in terms of A , B , and C :

$$\frac{x_2 - x_1}{2} = \sqrt{\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)} .$$

We have done it! The eqs. (4) and (5) now give us expressions for the roots x_1 and x_2 in terms of A , B , and C :

$$x_1 = -\frac{B}{2A} - \sqrt{\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)} ,$$

$$x_2 = -\frac{B}{2A} + \sqrt{\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)} .$$

We can algebraically arrange the expressions into the more customary form as in the theorem statement by multiplying both the numerator and denominator of the C/A term in the square root function by $4A$ and applying algebraic property (ii) to get

$$\sqrt{\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)} = \sqrt{\left(\frac{B^2}{4A^2} - \frac{4AC}{4A^2}\right)} = \sqrt{\frac{(B^2 - 4AC)}{4A^2}} = \frac{\sqrt{B^2 - 4AC}}{2A} .$$

Then x_1 and x_2 take their more familiar appearance:

$$x_1 = \frac{-B - \sqrt{B^2 - 4AC}}{2A} ,$$

$$x_2 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} .$$

Square roots of negative real numbers do not exist as real numbers, so $B^2 - 4AC$ under the square root sign must be non-negative. When $B^2 - 4AC$ is zero, the two roots merge into the value $-B/(2A)$. In Figure 1, panels (a) and (c) show quadratics with two real roots, and panels (b) and (d) show quadratics for which no real roots exist.